Extension of the pole decomposition for the multidimensional Burgers equation

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It is shown that the generalizations to more than one space dimension of the pole decomposition for the Burgers equation with finite viscosity ν and no force are of the form $\mathbf{u}=-2\nu\boldsymbol{\nabla}\ln P$, where the P's are explicitly known algebraic (or trigonometric) polynomials in the space variables with polynomial (or exponential) dependence on time. Such solutions have pole singularities on the complex algebraic varieties.

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I. INTRODUCTION

We are interested in the Burgers equation in $\mathbb{R}^n$ [1]

$$\mathbf{u}_t+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=\nu\nabla^2\mathbf{u},\quad \mathbf{u}=-\boldsymbol{\nabla}\Phi, \tag{1}$$

where $\mathbf{u}_t$ is the partial derivative of $\mathbf{u}$ with respect to time t and ν is viscosity ($\nu>0$). This equation is the simplest evolutionary dissipative equation, which is minimally (quadratically) nonlinear and enjoys translational and Galilean invariance. This simplicity and generality of the Eq. (1) explains its applicability for seemingly different processes occurring in a wide range of physical phenomena. Although originally this equation appeared as a model for Navier-Stokes turbulence [1], it is mostly used today in cosmology [2], polymer physics [3], and nonlinear acoustics [4]. (See Ref. [5] for a review.) Also this equation is very useful as a testing ground for numerical schemes in hydrodynamics [6]. These features make the Burgers equation important and attractive for physicists.

From the mathematical point of view the Burgers equation is also remarkable, for it is completely integrable [7], i.e., reducible to a *linear* problem after equivalent transformation. This follows directly from the Cole-Hopf transformation [8]:

$$\mathbf{u}=-2\nu\boldsymbol{\nabla}\ln\theta, \tag{2}$$

which maps (1) into the linear heat equation for the scalar field θ, namely,

$$\theta_t=\nu\nabla^2\theta, \tag{3}$$

from which the solution to the Burgers equation (1) can be obtained explicitly by quadrature.

With a few exceptions, completely integrable partial differential equations (PDEs) are two dimensional (one dimension for time and one dimension for space) [7]. There is a probable underlying mathematical reason, which hinders integrability in more than one spatial dimension: roughly speaking, it is related to the fact that polynomials with respect to more than one variable generally are not factorizable into nontrivial factors, while in the case of one variable they always are, by virtue of the main theorem of algebra. However, the Burgers equation (1) is integrable in arbitrary number of dimensions. This is obvious, because the Cole-Hopf transformation (2), which maps (1) to (3), is valid in $\mathbb{R}^n$ for an arbitrary natural n. Our goal here is to describe two classes of finite-dimensional exact solutions of the multidimensional Burgers equation (1), which are extensions of the "pole decomposition" of the (1+1)-dimensional Burgers equation [9,10].

The pole decomposition is a property of PDEs (or integro-PDEs) to have finite-dimensional solutions, whose degrees of freedom are movable singularities (poles) in the complex plane. Most, if not all, completely integrable models enjoy this property [7]. The most notable examples of pole decomposition in integrable systems can be found in Refs. [9,11–13]. The case of the (1+1)-dimensional Burgers equation (1) is discussed in Refs. [14,15]. Specifically, it admits "pole-decomposed" solutions in the form

$$\mathbf{u}(t,x)=-2\nu\sum_{k=1}^{N}\frac{1}{x-z_k(t)}, \tag{4}$$

where the poles constitute an N-dimensional dynamical system:

$$\frac{dz_l}{dt}=-2\nu\sum_{k\neq l}^{N}\frac{1}{z_l-z_k}. \tag{5}$$

The pole decomposition for Eq. (1) corresponds to solutions of Eq. (3) which are polynomial in the space variable. Note that existence of a pole decomposition for a nonlinear system does not imply its integrability. Indeed, there are known instances of nonlinear *nonintegrable* models, which also possess a pole decomposition. They include the two-dimensional Euler equation for ideal hydrodynamics [16], some models in plasma turbulence [17], some versions of the Sivashinsky equation for a flame propagation [18,19], and related combustion systems [20]. Both integrable and nonintegrable systems possessing a pole decomposition are of interest to physicists and mathematicians, for the pole dynamics reveals important physical trends and hidden mathematical structure underlying the model.

It should also be mentioned that the Burgers equation is dissipative, unlike almost all integrable systems, which are Hamiltonian [7]. The phase volume of closed dissipative systems shrinks with time, so that only a few degrees of freedom are really relevant in the long-time asymptotics, most of the initially existing degrees of freedom being eventually suppressed. Because a pole decomposition is an exact finite-dimensional reduction of the system with an infinite number of degrees of freedom, this explains why such a decomposition is especially instructive for dissipative models, such as the Burgers equation (1) [10], flame propagation [18–20], and viscous fingering (the Saffman-Taylor problem) [21].

Since multidimensional integrability is a far more difficult subject than the $(1+1)$-dimensional case, it is tempting to extend to higher-dimensions physical ideas adopted from interacting poles, by replacing poles by strings or by more general complex varieties. *A priori*, however, it seems impossible to extend Eq. (5) to non-pointlike objects, while keeping a finite number of degrees of freedom. However, we shall see that in higher dimensions there are still polynomial (algebraic and trigonometric) solutions to Eq. (3), which can be obtained explicitly. Observe that polynomials are factorizable in one dimension, whereas this is generally not the case in higher dimensions; the zeros of polynomials are then located on algebraic complex varieties which are generally *irreducible* [22].

Here, we will show that it is quite elementary to construct polynomial-based solutions to the multidimensional Burgers equation (1) with singularities on such irreducible varieties.

II. SOLUTIONS GENERATED BY POLYNOMIALS

We are looking for polynomial solutions to the heat equation

$$P(t,\mathbf{x})=\sum_{\mathbf{K}=\mathbf{0}}^{\mathbf{M}} a_{\mathbf{K}}(t)\prod_{l=1}^{n} x_l^{k_l}, \tag{6}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{K}=(k_1,k_2,\ldots,k_n)$, $\mathbf{0}=(0,0,\ldots,0)$, $\mathbf{M}=(m_1,m_2,\ldots,m_n)$, and $|\mathbf{M}|=\sum_{k=1}^{n}m_k$ is the degree of the polynomial. It is technically convenient to define new coefficients

$$b_{\mathbf{K}}=a_{\mathbf{K}}\prod_{l=1}^{n} k_l!. \tag{7}$$

The initial $(t=0)$ values of these coefficients are denoted by the superscript zero.

It is easily checked by substitution into the heat equation that the time dependence of the $b_{\mathbf{K}}$'s is

$$b_{\mathbf{K}}(t)=\sum_{\mathbf{P}=\mathbf{0}}^{\mathbf{M}'} b^0_{\mathbf{K}+2\mathbf{P}}\frac{(\nu t)^{p_1+p_2+\cdots+p_n}}{(p_1+p_2+\cdots+p_n)!}, \tag{8}$$

where $\mathbf{M}'=(m_1',m_2',\ldots,m_n')$, and m_l' is either m_l or m_l-1 depending on whether m_l-k_l in Eq. (8) is even or odd.

Using Eq. (2), we find that our polynomial solutions generate the following rational solutions to the Burgers equation:

$$u_p(t,\mathbf{x})=-2\nu\frac{\sum_{\mathbf{K}=\mathbf{0}}^{\mathbf{M}} a_{\mathbf{K}}(t)k_p\prod_{l=1}^{n} x_l^{k_l}}{x_p\sum_{\mathbf{K}=0}^{\mathbf{M}} a_{\mathbf{K}}(t)\prod_{l=1}^{n} x_l^{k_l}}, \tag{9}$$

where u_p is the pth component of the vector field $\mathbf{u}$. Another closely related class of solutions involves trigonometric polynomials

$$P(t,\mathbf{x})=\mathrm{Re}\sum_{\mathbf{K}=\mathbf{0}}^{\mathbf{M}} c_{\mathbf{K}}(t)\prod_{l=1}^{n} e^{ik_l x_l}, \tag{10}$$

with the same $\mathbf{K}$ and $\mathbf{M}$ as in Eq. (8). The time dependence of $c_{\mathbf{K}}$ is now given by

$$c_{\mathbf{K}}(t)=c^0_{\mathbf{K}}\exp\left[-\nu\left(\sum_{l=1}^{n} k_l^2\right)t\right]. \tag{11}$$

By Eq. (2) this generates the following solutions to the Burgers equation:

$$u_p(t,\mathbf{x})=2\nu\frac{\mathrm{Im}\sum_{\mathbf{K}=\mathbf{0}}^{\mathbf{M}} c_{\mathbf{K}}(t)k_p\prod_{l=1}^{n} e^{ik_l x_l}}{\mathrm{Re}\sum_{\mathbf{K}=\mathbf{0}}^{\mathbf{M}} c_{\mathbf{K}}(t)\prod_{l=1}^{n} e^{ik_l x_l}}. \tag{12}$$

Thus, we have shown that the Burgers equation (1) in $\mathbb{R}^n$ possesses exact solutions with a finite number of time-dependent parameters generated by the algebraic and trigonometric polynomial solutions of the heat equation in $\mathbb{R}^n$.

We observe that such solutions, contrary to the one-dimensional case, cannot, in general, be decomposed into a sum of separate simpler solutions. Indeed, this would correspond to having *at all-times* polynomial solutions of the heat equation which are factorized. Even if the initial polynomial is factorized, the time evolution will, in general, destroy the factorization. Recently, special solutions possessing the all-time factorization property were found by Leshchiner and one of the authors (M.M-W.). We do not yet know how broad is the class of such solutions.

A final remark concerns integrability and explicit characterization of singularities. Knowing explicitly the coefficients of the polynomial solution of the heat equation does not imply that we can explicitly describe the algebraic variety on which the polynomial vanishes. Even in one dimension, if we have a pole decomposition with more than four poles, we conjecture that Galois theory implies the following: given the initial position, in general, it is not possible to find the positions for all times by radicals.

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